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ON A REGULARIZED SOLUTION OF THE CAUCHY PROBLEM FOR MATRIX FACTORIZATIONS OF THE HELMHOLTZ EQUATION

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Abstract. In the paper the problem of regularization of the Cauchy problem for matrix factorizations of the Helmholtz equation in two-dimensional bounded domain of the type of a curvilinear triangle is considered. Using the Carleman matrix found an explicitly regularized solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in two-dimensional bounded domain.

Keywords: the Cauchy problem, regularization, factorization, regular solution, fundamental solution, domain of the type of a curvilinear triangle.

AMS Subject Classification: 35J46, 35J56.

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1 Introduction

It is known that the Cauchy problem for elliptic equations is unstable relatively small change in the data, i.e. incorrect (example Hadamard, (see for instance Adamar, 1978)). In unstable problems, the image of the operator is not is closed, therefore, the solvability condition can not be is written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on part of the boundary of the domain the solution is usually unique, the problem is solvable for everywhere dense a set of data, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than theory of solvability of Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov, N.N. Tarkhanov (see for instance (Carleman, 1926)).

The uniqueness of the solution follows from Holmgren's general theorem (see for instance Bers at al., 1966). The conditional stability of the problem follows from the work of A.N. Tikhonov (Tikhonov, 1963), if we restrict the class of possible solutions to a compactum.

In this paper we construct a family of vector-functions $U_{\sigma(\delta)}(x) = U(x, f_\delta)$ depending on a parameter σ and it is proved that, under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$; as $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ converges in the usual sense to a solution $U(x)$ at the point $x \in G$.

Following A.N. Tikhonov (Tikhonov, 1963), a family of vector-functions $U_{\sigma(\delta)}(x)$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem. For special domains, the problem of extending bounded analytic functions in the case when the data is specified exactly on a part of the boundary was considered by Carleman (Carleman, 1926). The researches of T. Carleman were continued by G.M. Goluzin and V.I. Krylov (see for instance Goluzin at al., 1933). A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in (see for instance Aizenberg, 1990). The use of the classical Green's formula for constructing a regularized solution of the Cauchy problem for the Laplace equation was proposed by Academician M.M. Lavrent'ev (Lavrent'ev, 1957; Lavrent'ev, 1962), in his famous monograph. Extending Lavrent'ev idea, Yarmukhamedov constructed the Carleman function for the Cauchy problem for the Laplace and Helmholtz equations (Yarmukhamedov, 1977; Yarmukhamedov, 1997; Yarmukhamedov, 2004). The Cauchy problem for the multidimensional Lame system is considered by O.I. Makhmudov and I.E. Niyozov (see for instance Makhmudov at al., 2006; Niyozov at al., 2014). Boundary problems for Helmholtz equation and the Cauchy problem for Dirac operators it was considered by A.A. Shlapunov (see for instance Shlapunov, 2011). The construction of the Carleman matrix for elliptic systems was carried out by Sh. Yarmukhamedov, N.N. Tarkhanov, A.A. Shlapunov, I.E. Niyozov, O.I. Makhmudov, and others.

The system considered in this paper was introduced by N.N. Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain. In many well-posed problems for a system of equations of elliptic type of the first order with constant coefficients, the factorizing operator of Helmholtz, the calculation of the value of the vector function on the whole boundary is inaccessible. Therefore, the problem of reconstructing, solving a system of equations of elliptic type of the first order with constant coefficients, the factorizing operator of Helmholtz (Juraev, 2012; Juraev, 2017a; Juraev, 2017b; Juraev, 2018a; Juraev, 2018b; Juraev, 2018c), is one of the topical problems in the theory of differential equations.

In this paper, we present an explicit formula for the approximate solution of the Cauchy problem for the matrix factorizations of the Helmholtz equation in a bounded region on the plane. The two-dimensional case requires special consideration, in contrast to three or more dimensions in many mathematical problems. Our formula for an approximate solution also includes the construction of a family of fundamental solutions for the Helmholtz operator on the plane. This family is parametrized by some entire function $K(w)$, the choice of which depends on the dimension of the space. This motivates a separate study of regularization formulas in flat domains and leads to improved estimates compared to the three-dimensional case.

Let $\mathbb{R}^2$ be a two-dimensional real Euclidean space,

$$x = (x_1, x_2) \in \mathbb{R}^2,\ y = (y_1, y_2) \in \mathbb{R}^2.$$

We introduce the following notation:

$$r = |y - x|,\ \alpha = |y_1 - x_1|, w = i\tau\sqrt{u^2 + \alpha^2} + \beta,\ w_0 = i\tau\alpha + \beta,$$

$$u \geq 0,\ \beta = \tau y_2, \tau = tg\frac{\pi}{2\rho},\ \rho > 1,\ \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)^T,$$

$$\frac{\partial}{\partial x} \to \xi^T,\ \xi^T = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \text{ is transposed vector for } \xi,$$

$$U(x) = (U_1(x), ..., U_n(x))^T,\ \ u^0 = (1, ..., 1) \in \mathbb{R}^n,\ \ n = 2^m,\ \ m = 2,$$

$$E(z) = \begin{Vmatrix} z_1 \dots 0 \\ \dots\dots\dots \\ 0 \dots z_n \end{Vmatrix} \text{ diagonal matrix, } z = (z_1, ..., z_n) \in \mathbb{R}^n.$$

$G_\rho \subset \mathbb{R}^2$ is a bounded simply connected domain whose boundary consists of segments of rays

$$|y_1| = \tau y_2,\ \ 0 < y_2 \leq y_0 < \infty,$$

with the beginning at zero and the arc S of a smooth curve lying inside the angle of width $\frac{\pi}{\rho}$, i.e. $\partial G_\rho = S \bigcup T,\ T = \partial G_\rho \backslash S$.

We assume that $(0, x_2) \in G_\rho,\ \ x_2 > 0$. G_ρ is called a domain of the type of a curvilinear triangle.

Let $D(\xi^T),\ (n \times n)-$be a matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the condition is satisfied:

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix $D(\xi^T)$, $\lambda-$a real number.

Let $x = (x_1, x_2) \in G_\rho,\ y = (y_1, y_2) \in \partial G_\rho$. We consider in the domain G_ρ a system of differential equations

$$D\left(\frac{\partial}{\partial x}\right)U(x) = 0, \tag{1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of differential operators is of the first order.

We denote by $A(G_\rho)$ the class of vector functions in the domain G_ρ, of continuous on $\overline{G}_\rho = G_\rho \bigcup \partial G_\rho$ and satisfying system (1).

2 The Cauchy problem and the construction of the Carleman function

Let $U(y) \in A(G_\rho)$ and

$$U(y)|_S = f(y),\ y \in S. \tag{2}$$

Here, $f(y)$ is a given continuous vector function on S.

It is required to restore the vector function $U(y)$ in the domain G_ρ, based on its values $f(y)$ on S.

If $U(y) \in A(G_\rho)$, then the following Cauchy type integral formula is true

$$U(x) = \int\limits_{\partial G_\rho} M(y, x)U(y)ds_y,\ \ x \in G_\rho, \tag{3}$$

where

$$M(y, x) = \left(E\left(-\frac{i}{4}H_0^{(1)}(\lambda r)u^0\right)D^*\left(\frac{\partial}{\partial y}\right)\right)D(t^T).$$

Here $t = (t_1, t_2)$ is the unit outward normal, carried out at the point y, the surface ∂G_ρ, $-\frac{i}{4}H_0^{(1)}(\lambda r)$ is the fundamental solution of the Helmholtz equation, defined through the Hankel function of the first kind (Aleksidze, 1991).

We denote by $K(w)$ is an entire function that takes real values for a real w ($w = u+iv,\ u,\ v-$ are real numbers) and satisfies the following conditions:

$$K(u) \neq 0,\ \sup_{v \geq 1}|v^p K^p(w)| = \mathrm{M}(u, p) < \infty,\ \ -\infty < u < \infty,\ p = 0, 1, 2. \tag{4}$$

The function $\Phi(y, x)$ at $y \neq x$ is defined by the following equation:

$$\Phi(y, x) = \frac{1}{2\pi K(x_2)}\int\limits_0^\infty \mathrm{Im}\frac{K(w)}{w - x_2}\frac{uI_0(\lambda u)}{\sqrt{u^2 + \alpha^2}}du. \tag{5}$$

Here $I_0(\lambda u) = J_0(i\lambda u)$ is the Bessel function of the first kind of zero order (see for instance Bers at al., 1966).

In the future, we use the following equalities

$$
\begin{gathered}
2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_1} = \frac{(y_1-x_1)\mathrm{Re}K(w_0) - sign(y_1-x_1)(y_2-x_2)\mathrm{Im}K(w_0)}{r^2} - \\
-(y_1-x_1)\lambda\int\limits_0^\infty \frac{\sqrt{u^2+\alpha^2}\mathrm{Re}K(w)-(y_2-x_2)\mathrm{Im}K(w)}{u^2+r^2}\cdot\frac{I_1(\lambda u)du}{\sqrt{u^2+\alpha^2}},\ y\neq x, \\
w_0 = i\,|y_1-x_1| + y_2,\ I_1(\lambda u) = I_0^{'}(\lambda u)
\end{gathered} \tag{6}
$$

and

$$
\begin{gathered}
2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_2} = \frac{(y_2-x_2)\mathrm{Re}K(w_0) - (y_1-x_1)\mathrm{Im}K(w_0)}{r^2} - \\
-\lambda\int\limits_0^\infty \frac{(y_2-x_2)\mathrm{Re}K(w)-\sqrt{u^2+\alpha^2}\mathrm{Im}K(w)}{u^2+r^2}I_1(\lambda u)du,\ y_1\neq x_1,
\end{gathered} \tag{7}
$$

which are obtained from (5).

In the formula (5), choosing

$$
K(w) = E_\rho(\sigma^{1/\rho}w),\ K(x_2) = E_\rho(\sigma^{1/\rho}\gamma),\ \gamma=\tau x_2,\ \sigma>0, \tag{8}
$$

we get

$$
\begin{gathered}
\Phi_\sigma(y,x) = \frac{E_\rho(\sigma^{1/\rho}\gamma)}{2\pi}\int\limits_0^\infty \mathrm{Im}\frac{E_\rho(\sigma^{1/\rho}w)}{w-x_2}\frac{u I_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du, \\
\sigma\geq\lambda+\sigma_0,\ \sigma_0>0.
\end{gathered} \tag{9}
$$

Here $E_\rho(\sigma^{1/\rho}w)$ is the entire Mittag-Leffler function.

Accordingly, in formulas (6) - (7), using equalities (8) and choosing

$$
K(w_0) = E_\rho(\sigma^{1/\rho}w_0),\ \ w_0 = i\tau\alpha+\beta,\ \beta=\tau y_2,\ \sigma>0, \tag{10}
$$

we obtain the following analogous formulas

$$
\begin{gathered}
2\pi E_\rho(\sigma^{1/\rho}\gamma)\frac{\partial\Phi_\sigma(y,x)}{\partial y_1} = \\
= \frac{(y_1-x_1)\mathrm{Re}\exp(E_\rho(\sigma^{1/\rho}\omega_0) + sign(y_1-x_1)(y_2-x_2)\mathrm{Im}\exp(E_\rho(\sigma^{1/\rho}\omega_0)}{r^2} - \\
-(y_1-x_1)\lambda\int\limits_0^\infty \frac{\sqrt{u^2+\alpha^2}\mathrm{Re}\exp(E_p(\sigma^{1/\rho}\omega)-(y_2-x_2)\mathrm{Im}\exp(E_p(\sigma^{1/\rho}\omega)}{u^2+r^2}\cdot\frac{I_1(\lambda u)du}{\sqrt{u^2+\alpha^2}}, \\
y\neq x,\quad \omega_0 = i\tau\alpha+\beta,\quad \beta=\tau y_2,
\end{gathered} \tag{11}
$$

and

$$
\begin{gathered}
2\pi E_\rho(\sigma^{1/\rho}\gamma)\frac{\partial\Phi_\sigma(y,x)}{\partial y_2} = \frac{(y_2-x_2)\mathrm{Re}\exp(E_\rho(\sigma^{1/\rho}w_0) + (y_1-x_1)\mathrm{Im}\exp(E_\rho(\sigma^{1/\rho}w_0)}{r^2} - \\
-\lambda\int\limits_0^\infty \frac{(y_2-x_2)\mathrm{Re}\exp(E_\rho(\sigma^{1/\rho}w)-\sqrt{u^2+\alpha^2}\mathrm{Im}\exp(E_\rho(\sigma^{1/\rho}w)}{u^2+r^2}I_1(\lambda u)du,\ y_1\neq x_1, \\
w_0 = i\tau\alpha+\beta,\ \beta=\tau y_2.
\end{gathered} \tag{12}
$$

Lemma 1. *Let* $x = (x_1, x_2) \in G_\rho,\ y \neq x,\ \sigma \geq \lambda + \sigma_0,\ \sigma_0 > 0$, *then*
1) at $\beta \leq \alpha$ *inequalities are satisfied*

$$|\Phi_\sigma(y, x)| \leq C(\rho) E_\rho^{-1}(\sigma^{1/\rho} w)\sigma \exp(-\sigma\gamma^\rho),\ \sigma > 1,\ x \in G_\rho, \tag{13}$$

$$\left|\frac{\partial\Phi(y, x)}{\partial y_1}\right| \leq C(\rho) E_\rho^{-1}(\sigma^{1/\rho} w)\sigma \exp(-\sigma\gamma^\rho),\ \sigma > 1,\ x \in G_\rho, \tag{14}$$

$$\left|\frac{\partial\Phi(y, x)}{\partial y_2}\right| \leq C(\rho) E_\rho^{-1}(\sigma^{1/\rho} w)\sigma \exp(-\sigma\gamma^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{15}$$

2) at $\beta > \alpha$ *inequalities are satisfied*

$$|\Phi_\sigma(y, x)| \leq C(\rho) E_\rho^{-1}(\sigma^{1/\rho} w)\sigma \exp(-\sigma \mathrm{Re}\, w_0^\rho),\ \sigma > 1,\ x \in G_\rho, \tag{16}$$

$$\left|\frac{\partial\Phi(y, x)}{\partial y_1}\right| \leq C(\rho) E_\rho^{-1}(\sigma^{1/\rho} w)\sigma \exp(-\sigma \mathrm{Re}\, w_0^\rho),\ \sigma > 1,\ x \in G_\rho, \tag{17}$$

$$\left|\frac{\partial\Phi(y, x)}{\partial y_2}\right| \leq C(\rho) E_\rho^{-1}(\sigma^{1/\rho} w)\sigma \exp(-\sigma \mathrm{Re}\, w_0^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{18}$$

Here $C(\rho)$ *is the function depending on* ρ.

Recall the basic properties of the Mittag-Leffler function. The entire function of Mittag-Leffler is defined by a series

$$\sum_{n=1}^{\infty} \frac{w^n}{\Gamma(1 + \rho^{-1} n)} = E_\rho(w),\ w = u + iv,$$

where $\Gamma(s)$ is the Euler gamma function.

We denote by $\gamma_\varepsilon(\beta_0)(\varepsilon > 0,\ 0 < \beta_0 < \pi)$ the contour in the complex plane ζ, run in the direction of non-decreasing $\arg\zeta$ and consisting of the following parts:
1. The beam $\arg\zeta = -\beta_0,\ |\zeta| \geq \varepsilon$;
2. The arc $-\beta_0 < \arg\zeta < \beta_0$ of circle $|\zeta| = \varepsilon$;
3. The beam $\arg\zeta = \beta_0,\ |\zeta| \geq \varepsilon$.

The contour $\gamma_\varepsilon(\beta_0)$ divides the plane ζ into two unbounded simply connected domains G_ρ^- and G_ρ^+ lying to the left and to the right of $\gamma_\varepsilon(\beta_0)$, respectively.

Let $\rho > 1,\ \frac{\pi}{2\rho} < \beta_0 < \frac{\pi}{\rho}$.

Denote
$$\psi_\rho(w) = \frac{1}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{\exp(\zeta^\rho)}{\zeta - w} d\zeta. \tag{19}$$

Then the following integral representations are valid:

$$E_\rho(w) = \psi_\rho(w), \quad z \in G_\rho^-, \tag{20}$$

$$E_\rho(w) = \rho \exp(w^\rho) + \psi_\rho(w), \quad z \in G_\rho^+. \tag{21}$$

From these formulas we find

$$\left.\begin{aligned} &|E_\rho(w)| \leq \rho \exp(\mathrm{Re}\, w^\rho) + |\psi_\rho(w)|,\ \ |\arg w| \leq \tfrac{\pi}{2\rho} + \eta_0, \\ &|E_\rho(w)| \leq |\psi_\rho(w)|,\ \tfrac{\pi}{2\rho} + \eta_0 \leq |\arg w| \leq \pi,\ \ \eta_0 > 0 \end{aligned}\right\} \tag{22}$$

$$|\psi_\rho(w)| \leq \frac{M}{1+|w|},\ M = const \tag{23}$$

$$E_\rho(w) \approx \rho \exp(w^\rho),\ \ w > 0,\ \ w \to \infty, \tag{24}$$

Further, since $E_\rho(w)$ is real with real w, then

$$\operatorname{Re}\psi_\rho(w) = \frac{\rho}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{2\zeta - \operatorname{Re} w}{(\zeta - w)\zeta - \overline{w})} \exp(\zeta^\rho) d\zeta,$$

$$\operatorname{Im}\psi_\rho(w) = \frac{\rho \operatorname{Im}(w)}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{\exp(\zeta^\rho)}{(\zeta - w)\zeta - \overline{w})} d\zeta,$$

The information given here concerning the function $E_\rho(w)$ is taken from (see for instance Niyozov at al., 2014; Juraev, 2018).

Formula (3) is true if instead of $-\frac{i}{4}H_0^{(1)}(\lambda r)$ we substitute the function

$$\Phi_\sigma(y, x) = -\frac{i}{4}H_0^{(1)}(\lambda r) + g_\sigma(y, x), \tag{25}$$

where $g_\sigma(y, x)$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then the integral formula has the form:

$$U(x) = \int\limits_{\partial G_\rho} N_\sigma(y, x) U(y) ds_y,\ \ x \in G_\rho, \tag{26}$$

where

$$N_\sigma(y, x) = \left(E\left(\Phi_\sigma(y, x) u^0\right) D^*\left(\frac{\partial}{\partial y}\right)\right) D(t^T).$$

For a fixed $x \in G_\rho$ we denote by S^* the part of S on which $\beta \geq \alpha$. If $x \in G_\rho$, then $S = S^*$ (in this case, $\beta = \tau y_2$ and the inequality $\beta \geq \alpha$ means that y lies inside or on the surface cone).

Theorem 1. *Let $U(y) \in A(G_\rho)$ satisfy the inequality*

$$|U(y)| \leq 1,\ y \in T = \partial G_\rho \backslash S^*, \tag{27}$$

If

$$U_\sigma(x) = \int\limits_{S^*} N_\sigma(y, x) U(y) ds_y,\ \ x \in G_\rho, \tag{28}$$

then we have the estimate

$$|U(x) - U_\sigma(x)| \leq C_\rho(\lambda, x)\sigma \exp(-\sigma\gamma^\rho),\ \ \sigma > 1,\ x \in G_\rho. \tag{29}$$

Here and below functions bounded on compact subsets of the domain C_ρ, we denote by $C_\rho(\lambda, x)$.

Proof. Using integral formula (26) and the equality (28), we get

$$U(x) = U_\sigma(x) + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x) U(y) ds_y,\ x \in G_\rho.$$

Taking into account inequality (27), we estimate the following

$$|U(x) - U_\sigma(x)| \le \left| \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y,x) U(y) ds_y \right| \le \int\limits_{\partial G_\rho \backslash S^*} |N_\sigma(y,x)|\, ds_y,\ \ x \in G_\rho. \tag{30}$$

Let us estimate the integrals

$$\int\limits_{\partial G_\rho \backslash S^*} |\Phi_\sigma(y,x)|\, dy_y, \qquad \int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_1} \right| ds_y$$

and

$$\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_2} \right| ds_y.$$

Let $\sigma > 0$. Separating the imaginary part of equality (9), we get

$$\begin{aligned} \Phi_\sigma(y,x) = \tfrac{E_\rho(\sigma^{1/\rho}\gamma)}{2\pi} \Bigg[\int\limits_0^\infty \frac{(y_2 - x_2)\mathrm{Im} E_\rho(\sigma^{1/\rho} w)}{u^2 + r^2}\ \tfrac{u\, I_0(\lambda u)}{\sqrt{u^2+\alpha^2}} - \\ - \int\limits_0^\infty \frac{u \mathrm{Re}\, E_\rho(\sigma^{1/\rho} w)}{u^2 + r^2} I_0(\lambda u) du \Bigg],\ y \ne x,\ x_2 > 0. \end{aligned} \tag{31}$$

Given (31) and inequality

$$I_0(\lambda u) \le \exp(\lambda u), \tag{32}$$

we have

$$\int\limits_{\partial G_\rho \backslash S^*} |\Phi_\sigma(y,x)|\, dy_1 \le C_\rho(\lambda, x) \sigma \exp(-\sigma \gamma^\rho),\ \ \sigma > 1,\ x \in G_\rho. \tag{33}$$

To estimate the integrals $\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_1} \right| ds_y$ and $\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_2} \right| ds_y$, we use equalities (11) and (12).

Considering equality (11) and inequality

$$I_1(\lambda u) \le \lambda u \exp(\lambda u), \tag{34}$$

we get

$$\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_1} \right| ds_y \le C_\rho(\lambda, x) \sigma \exp(-\sigma \gamma^\rho),\ \ \sigma > 1,\ x \in G_\rho. \tag{35}$$

Similarly, taking into account equality (12) and inequality (34), we estimate the following integral

$$\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_2} \right| ds_y \le C_\rho(\lambda, x) \sigma \exp(-\sigma \gamma^\rho),\ \ \sigma > 1,\ x \in G_\rho. \tag{36}$$

From inequalities (33), (35) and (36), we obtain (29). Theorem 1 is proved. □

Corollary 1. The limiting equality

$$\lim_{\sigma \to \infty} U_\sigma(x) = U(x),$$

holds uniformly on each compact set in the domain G_ρ.

Theorem 2. *Suppose that $U(y) \in A(G_\rho)$ satisfies condition (27) and, on a smooth curve S the inequality*

$$|U(y)| \leq \delta,\ 0 < \delta < 1, \tag{37}$$

then the estimate is correct

$$|U(x)| \leq C_\rho(\lambda, x)\sigma\delta^{\left(\frac{\gamma}{R}\right)^\rho},\ \sigma > 1,\ x \in G_\rho. \tag{38}$$

Here is $R^\rho = \max\limits_{y\in S} \operatorname{Re} w_0$.

Proof. From (26) and equality (28) with, we have

$$U(x) = \int\limits_{S^*} N_\sigma(y, x)U(y)ds_y + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x)U(y)ds_y. \tag{39}$$

We estimate the following

$$|U(x)| \leq \left|\int\limits_{S^*} N_\sigma(y, x)U(y)ds_y\right| + \left|\int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x)U(y)ds_y\right|,\ x \in G_\rho. \tag{40}$$

Taking into account inequality (37), we estimate the first term inequality (40).

$$\begin{aligned}\left|\int\limits_{S^*} N_\sigma(y, x)U(y)ds_y\right| \leq \int\limits_{S^*} |N_\sigma(y, x)|\,|U(y)|\,ds_y \leq \\ \leq \delta \int\limits_{S^*} |N_\sigma(y, x)|\,ds_y,\ x \in G_\rho.\end{aligned} \tag{41}$$

To do this, we estimate the integrals

$$\int\limits_{S^*} |\Phi_\sigma(y, x)|\,dy_y,\quad \int\limits_{S^*} \left|\frac{\partial\Phi_\sigma(y, x)}{\partial y_1}\right| ds_y$$

and

$$\int\limits_{S^*} \left|\frac{\partial\Phi_\sigma(y, x)}{\partial y_2}\right| ds_y$$

on a smooth curve S.

Considering equality (31) and inequality (32), we have

$$\int\limits_{S^*} |\Phi_\sigma(y, x)|\,ds_y \leq C_\rho(\lambda, x)\sigma \exp\sigma(\tau^\rho R^\rho - \tau^\rho x_2^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{42}$$

Using equation (11) and inequality (34), we have

$$\int\limits_{S^*} \left|\frac{\partial\Phi_\sigma(y, x)}{\partial y_1}\right| ds_y \leq C_\rho(\lambda, x)\sigma \exp\sigma(\tau^\rho R^\rho - \tau^\rho x_2^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{43}$$

Similarly, using equality (12) and inequality (34), we get

$$\int\limits_{S^*} \left|\frac{\partial\Phi_\sigma(y, x)}{\partial y_2}\right| ds_y \leq C_\rho(\lambda, x)\sigma \exp\sigma(\tau^\rho R^\rho - \tau^\rho x_2^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{44}$$

From (41) - (44), we get

$$\left| \int\limits_{S^*} N_\sigma(y,x)U(y)ds_y \right| \leq C_\rho(\lambda,x)\sigma\delta \exp\sigma(\tau^\rho R^\rho - \tau^\rho x_2^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{45}$$

The following is known

$$\left| \int\limits_{\partial G_\rho \setminus S^*} N_\sigma(y,x)U(y)ds_y \right| \leq C_\rho(\lambda,x)\sigma \exp(-\sigma\gamma^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{46}$$

Now, considering (45) - (46), we have

$$|U(x)| \leq \frac{C_\rho(\lambda,x)\sigma}{2}(\delta \exp(\sigma\tau^\rho R^\rho) + 1)\exp(-\sigma\tau^\rho x_2^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{47}$$

Choosing σ from equality

$$\sigma = \frac{1}{\tau^\rho R^\rho} \ln \frac{1}{\delta}, \tag{48}$$

we obtain inequality (38). Theorem 2 is proved. □

Let $U(y) \in A(G_\rho)$ and instead of $U(y)$ on S, its approximation $f_\delta(y)$ be set, respectively, with an error $0 < \delta < 1$, $\max\limits_S |U(y) - f_\delta(y)| \leq \delta$.

Set

$$U_{\sigma(\delta)}(x) = \int\limits_{S^*} N_\sigma(y,x)f_\delta(y)ds_y,\ x \in G_\rho. \tag{49}$$

The following theorem takes place

Theorem 3. *Let $U(y) \in A(G_\rho)$ on the entire boundary of ∂G_ρ satisfy the boundary condition* (27). *Then we have the estimate*

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \leq C_\rho(\lambda,x)\sigma\delta^{\left(\frac{\gamma}{R}\right)^\rho},\ \sigma > 1,\ x \in G_\rho. \tag{50}$$

Proof. From the integral formula (26) and equality (49), we have

$$U(x) - U_{\sigma(\delta)}(x) = \int\limits_{S^*} N_\sigma(y,x)\left\{U(y) - f_\delta(y)\right\} ds_y + \int\limits_{\partial G_\rho \setminus S^*} N_\sigma(y,x)U(y)\, ds_y.$$

Now, repeating the proofs of Theorems 1 and 2, we obtain

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \leq \frac{C_\rho(\lambda,x)\sigma}{2}(\delta \exp(\sigma\tau^\rho R^\rho) + 1)\exp(-\sigma\tau^\rho x_2^\rho),\ \sigma > 1,\ x \in G_\rho.$$

From here, choosing σ from equality (48), we obtain (50). Theorem 3 is proved. □

Corollary 2. The limiting equality

$$\lim_{\delta \to 0} U_{\sigma(\delta)}(x) = U(x),$$

holds uniformly on each compact set in the domain G_ρ.

3 Conclusion

In conclusion, we can say that on the basis of previous works (Juraev, 2012; Juraev, 2017a; Juraev, 2017b; Juraev, 2018a; Juraev, 2018b; Juraev, 2018c), we constructed the Carleman matrix, and using this function, we found a regularized solution of the Cauchy problem for the matrix factorizations of the Helmholtz equation for a two-dimensional bounded domain in an explicit form. Thus, the functional $U_{\sigma(\delta)}(x)$ determines the regularization of the solution of problem (1)-(2).

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References

Adamar, J. (1978). *The Cauchy problem for linear partial differential equations of hyperbolic type*. Moscow, Nauka (in Russian).

Aizenberg, L.A. (1990). *Carleman's formulas in complex analysis*. Novosibirsk, Nauka (in Russian).

Aleksidze, M.A. (1991). *Fundamental functions in approximate solutions of boundary value problems*. Moscow, Nauka (in Russian).

Arbuzov, E.V., Bukhgeim, A.L. (2006). The Carleman formula for the Helmholtz equation. *Sib. Math. Journal.*, *47*(3), 518-526 (in Russian).

Bers, A., John, F., Shekhter, M. (1966). *Partial Differential Equations*. Moscow, Mir (in Russian).

Carleman, T. (1926). *Les fonctions quasi analytiques*. Gautier-Villars et Cie., Paris.

Goluzin, G.M., Krylov, V.M. (1933). The generalized Carleman formula and its application to the analytic continuation of functions. *Math. Sbornik.*, *40*(2), 144-149 (in Russian).

Juraev, D.A. (2012). The construction of the fundamental solution of the Helmholtz equation. *Reports of the Academy of Sciences of the Republic of Uzbekistan, 2*, 14-17 (in Russian).

Juraev, D.A. (2017a). The Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain. *Sib. Electron. Mat. Izv., 14*, 752-764 (in Russian).

Juraev, D.A. (2017b). Cauchy problem for matrix factorizations of the Helmholtz equation. *Ukrainian Mathematical Journal, 69*(10), 1364-1371 (in Russian).

Juraev, D.A. (2018a). On the Cauchy problem for matrix factorizations of the Helmholtz equation in a bounded domain. *Sib. Electron. Mat. Izv., 15*, 11–20 (in Russian).

Juraev, D.A. (2018b). The Cauchy problem for matrix factorizations of the Helmholtz equation in $\mathbb{R}^3$. *Journal of Universal Mathematics, 1*(3), 312-319.

Juraev, D.A. (2018c). On the Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain in $\mathbb{R}^2$. *Sib. Electron. Mat. Izv., 15*, 1865-1877 (in Russian).

Lavrent'ev, M.M. (1957). On the Cauchy problem for second-order linear elliptic equations. *Dokl. Akad. Nauk SSSR*, *112*(2), 195-197 (in Russian).

Lavrent'ev, M.M. (1962). *On some ill-posed problems of mathematical physics.* Novosibirsk, Nauka (in Russian).

Makhmudov, O.I., Niyozov, I.E. (2006). On the Cauchy problem for a multidimensional system of Lame equation. *Izv. Vuz. Mat,. 4*, 41-50 (in Russian).

Niyozov, I.E., Makhmudov, O.I. (2014). The Cauchy problem of the moment elasticity theory in $\mathbb{R}^m$. *Izv. Vuz. Mat., 58*(2), 24-30 (in Russian).

Shlapunov A.A. (2011). Boundary problems for Helmholtz equation and the Cauchy problem for Dirac operators. *J. Sib. Fed. Univ. Math. Phys., 4*(2), 217-228.

Tarkhanov, N.N. (1985). On the Carleman matrix for elliptic systems. *Dokl. Akad. Nauk SSSR, 284*(2), 294-297.

Tarkhanov, N.N. (1995). *The Cauchy problem for solutions of elliptic equations.* Akad. Verl., Berlin, V. 7.

Tikhonov, A.N. (1963). On the solution of ill-posed problems and the regularization method. *Dokl. Akad. Nauk SSSR, 151*(3), 501-504 (in Russian).

Yarmukhamedov, Sh. (1977). On the Cauchy problem for the Laplace equation. *Dokl. Akad. Nauk SSSR, 235*(2), 281-283 (in Russian).

Yarmukhamedov, Sh. (1997). On the extension of the solution of the Helmholtz equation. *Dokl. Ross. Akad. Nauk, 357*(3), 320-323 (in Russian).

Yarmukhamedov, Sh. (2004). The Carleman function and the Cauchy problem for the Laplace equation. *Sibirsk. Math. Journal., 45*(3), 702-713 (in Russian).